

**ANALYTICAL SOLUTIONS OF THE KLEIN - GORDON EQUATION WITH
THE WOODS-SAXON POTENTIAL FOR ARBITRARY l - STATE****V.H.BADALOV, H.I.AHMADOV^{*}, S.V.BADALOV^{**}*****Institute for Physical Problems,*******Department of Mathematical Physics********Faculty of Physics, Baku State University*****e-mail: badalovvatan@yahoo.com**

In the paper the analytical solution of the Klein - Gordon equation for the Woods - Saxon potential is presented. In our calculations we have applied the Nikiforov-Uvarov method by using the Pekeris approximation to the centrifugal potential for arbitrary l states. The exact bound state energy eigenvalues and the corresponding eigenfunctions are obtained for a particle bound on the various values of the quantum numbers n_r and l .

Key words: Klein - Gordon equation, Nikiforov-Uvarov method, Pekeris approximation, exact solutions

1. INTRODUCTION

An analytical solution of the radial part of the Klein - Gordon (KG) equation is of high importance for spinless relativistic quantum mechanics, because the wave function contains all necessary information on full description of a quantum system. There are only few potentials for which the radial part of the Klein - Gordon equation can be solved explicitly for all n_r and l . So far, many methods were developed, such as supersymmetry (SUSY) [1,2] and Pekeris approximation [3-6], to solve radial part of the Klein - Gordon equation exactly or quasi-exactly for $l \neq 0$ within these potentials.

The one - dimensional Klein - Gordon equation is investigated for the PT - symmetric generalized Woods - Saxon (WS) potential [7] and Hulthen potential [8] and is solved by using the Nikiforov - Uvarov method, which is based on solving the second - order linear differential equation by reduction to a generalized equation of hypergeometric type.

The radial part of the Klein - Gordon equation for the Woods - Saxon potential [9] can not be solved exactly for $l \neq 0$. It is well known, that the Woods - Saxon potential is one of the important short - range potentials in physics. This potential was applied to numerous problems, in nuclear and particle physics, atomic physics, condensed matter and chemical physics. Therefore, it would be interesting and important to solve the radial Klein - Gordon equation for Woods-Saxon potential for

$l \neq 0$, since it has been extensively used to describe the bound and continuum states of the interacting systems.

In this work, we solve the radial Klein - Gordon equation for the standard Woods - Saxon potential using NU method [10], and obtain the energy eigenvalues and corresponding eigenfunctions for arbitrary l states.

2. NIKIFOROV-UVAROV METHOD

The Nikiforov - Uvarov (NU) method is based on the solutions of general second order linear equations with special orthogonal functions. It has been extensively used to solve the non-relativistic Schrödinger equation and other Schrödinger - like equations. The one - dimensional Schrödinger equation or similar second - order differential equations can be written using NU method in the following form:

$$\psi''(z) + \frac{\tilde{\tau}(z)}{\sigma(z)}\psi'(z) + \frac{\tilde{\sigma}(z)}{\sigma^2(z)}\psi(z) = 0, \quad (2.1)$$

where $\sigma(z)$ and $\tilde{\sigma}(z)$ are polynomials, at most second-degree, and $\tilde{\tau}(z)$ is a first - degree polynomial.

Using in Eq(2.1) the transformation

$$\psi(z) = \Phi(z)y(z) \quad (2.2)$$

one reduces it to the hypergeometric type equation

$$\sigma(z)y'' + \tau(z)y' + \lambda y = 0. \quad (2.3)$$

The function $\Phi(z)$ is defined as the logarithmic derivative [10]

$$\frac{\Phi'(z)}{\Phi(z)} = \frac{\pi(z)}{\sigma(z)}, \quad (2.4)$$

where $\pi(z)$ is at most the first - degree polynomial.

The another part of $\psi(z)$, namely $y(z)$ is the hypergeometric - type function, that for fixed n is given by the Rodriguez relation

$$y_n(z) = \frac{B_n}{\rho(z)} \frac{d^n}{dz^n} [\sigma^n(z)\rho(z)], \quad (2.5)$$

where B_n is the normalization constant and the weight function $\rho(z)$ must satisfy the condition [10]

$$\frac{d}{dz}(\sigma(z)\rho(z)) = \tau(z)\rho(z), \quad (2.6)$$

with $\tau(z) = \tilde{\tau}(z) + 2\pi(z)$.

For accomplishment of the conditions imposed on function $\rho(z)$, the classical orthogonal polynomials, it is necessary, that polynomial $\tau(z)$ becomes equal to zero in some point of an interval (a, b) and derivative of this polynomial for this interval at $\sigma(z) > 0$ will be negative, i.e. $\tau'(z) < 0$.

The function $\pi(z)$ and the parameter λ required for this method are defined as follows

$$\pi(z) = \frac{\sigma' - \tilde{\tau}}{2} \pm \sqrt{\left(\frac{\sigma' - \tilde{\tau}}{2}\right)^2 - \tilde{\sigma} + k\sigma}, \quad (2.7)$$

$$\lambda = k + \pi'(z). \quad (2.8)$$

On the other hand, in order to find the value of k , the expression under the square root must be the square of a polynomial. This is possible only if its discriminant is zero. Thus, the new eigenvalue equation for the Schrödinger equation is [10]

$$\lambda = \lambda_n = -n\tau' - \frac{n(n-1)}{2}\sigma'', \quad (n = 0, 1, 2, \dots). \quad (2.9)$$

After comparison of Eq.(2.8) with Eq.(2.9), we obtain the energy eigenvalues.

3. SOLUTIONS OF THE KLEIN - GORDON EQUATION WITH THE WOODS-SAXON POTENTIAL

The standard Woods-Saxon potential [9] is defined by

$$V(r) = -\frac{V_0}{1 + e^{\frac{r-R_0}{a}}} \quad (a \ll R_0), \quad (3.1)$$

where V_0 is the potential depth, R_0 is the width of the potential or the nuclear radius and the parameter a is the thickness of the superficial layer inside of which the potential falls from value $V = 0$ outside of a nucleus up to value $V = -V_0$ inside a nucleus. At $a = 0$ one gets the simple potential well with jump of potential on the surface of a nucleus.

In the spherical coordinates, the stationary Klein - Gordon equation with Woods-Saxon potential is

$$-\hbar^2 c^2 \left\{ \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \left[\frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right] \right\} \psi = \left[(E - V(r))^2 - m_0^2 c^4 \right] \psi$$

where m_0 is the rest mass of a scalar particle.

The terms in the square brackets with the overall minus sign are the dimensionless angular momentum squared operator \hat{L}^2 . Defining $\psi(r, \theta, \varphi) = R(r)Y(\theta, \varphi)$, we obtain the radial part of the Klein - Gordon equation [11]

$$\frac{d^2 R(r)}{dr^2} + \frac{2}{r} \frac{dR(r)}{dr} + \left[\frac{(E - V)^2 - m_0^2 c^4}{\hbar^2 c^2} - \frac{l(l+1)}{r^2} \right] R(r) = 0, \quad (0 \leq r < \infty), \quad (3.2)$$

where l is the angular momentum quantum number.

After introducing the new function $u(r) = rR(r)$, Eq.(3.2) takes the form

$$\frac{d^2 u(r)}{dr^2} + \left[\frac{(E - V)^2 - m_0^2 c^4}{\hbar^2 c^2} - \frac{l(l+1)}{r^2} \right] u(r) = 0. \quad (3.3)$$

The Eq.(3.3) has the same form as the equation for a particle in one dimension, except for two important differences. First, there is a repulsive effective potential propor-

tional to the eigenvalue of $\hbar^2 l(l+1)$. Second, the radial function must satisfy the boundary conditions $u(0) = 0$ and $u(\infty) = 0$.

If in Eq.(3.1) to introduce the notations

$$x = \frac{r-R_0}{R_0}, \quad \alpha = \frac{R_0}{a},$$

then the Woods - Saxon potential is given by the expression

$$V_{WS} = -\frac{V_0}{1 + e^{\alpha x}}.$$

It is known that KG equation cannot be solved exactly for this potential at the value $l \neq 0$ using the standard methods as SUSY or NU. From Eq.(3.3) it is seen, that the centrifugal potential $V_l(r)$ is inverse square potentials, which cannot be solved analytically. Therefore, in order to solve this problem we can take the most widely used and convenient for our purposes Pekeris approximation. This approximation is based on the expansion of the centrifugal barrier in a series of exponentials depending on the internuclear distance, taking into account terms up to second order, so that the l dependent potential preserves the original form. It should be pointed out, however, that this approximation is valid only for low vibrational energy states. By changing the coordinates $x = \frac{r-R_0}{R_0}$ or $r = R_0(1+x)$, the centrifugal potential is expanded in the

Taylor series around the point $x = 0$ ($r = R_0$)

$$V_l(r) = \frac{\hbar^2 l(l+1)}{2m_0 r^2} = \frac{\hbar^2 l(l+1)}{2m_0 R_0^2} \frac{1}{(1+x)^2} = \delta(1 - 2x + 3x^2 - 4x^3 + \dots), \quad (3.4)$$

where $\delta = \frac{\hbar^2 l(l+1)}{2m_0 R_0^2}$.

According to the Pekeris approximation, we shall replace potential $V_l(r)$ with expression

$$\tilde{V}_l(r) = \delta \left(C_0 + \frac{C_1}{1 + e^{\alpha x}} + \frac{C_2}{(1 + e^{\alpha x})^2} \right). \quad (3.5)$$

In order to define the constants C_0, C_1 and C_2 , we also expand this potential in the Taylor series around the point $x = 0$ ($r = R_0$)

$$\tilde{V}_l(x) = \delta \left[\left(C_0 + \frac{C_1}{2} + \frac{C_2}{4} \right) - \frac{\alpha}{4} (C_1 + C_2)x + \frac{\alpha^2}{16} x^2 + \frac{\alpha^3}{48} (C_1 + C_2)x^3 - \frac{\alpha^4}{96} x^4 + \dots \right]. \quad (3.6)$$

Comparing equal powers of x in Eqs.(3.4) and (3.6), we obtain the constants C_0, C_1 and C_2

$$C_0 = 1 - \frac{4}{\alpha} + \frac{12}{\alpha^2}, \quad C_1 = \frac{8}{\alpha} - \frac{48}{\alpha^2}, \quad C_2 = \frac{48}{\alpha^2}.$$

Instead of solving the radial part of the Klein - Gordon equation for the centrifugal potential $V_l(r)$ given by Eq.(3.4), we now solve the radial part of the Klein - Gordon equation for the new centrifugal potential $\tilde{V}_l(r)$ given by Eq.(3.5) and obtained using the Pekeris approximation. Having inserted this new centrifugal potential into Eq.(3.3), we obtain

$$\frac{d^2u}{dr^2} + \left[\left(\frac{E^2 - m_0^2 c^4}{\hbar^2 c^2} - \frac{l(l+1)C_0}{\alpha^2 a^2} \right) + \frac{\left(\frac{2EV_0}{\hbar^2 c^2} - \frac{l(l+1)C_1}{\alpha^2 a^2} \right)}{1 + e^{\frac{r-R_0}{a}}} + \frac{\left(\frac{V_0^2}{\hbar^2 c^2} - \frac{l(l+1)C_2}{\alpha^2 a^2} \right)}{\left(1 + e^{\frac{r-R_0}{a}} \right)^2} \right] u = 0. \quad (3.7)$$

We use the following dimensionless notations

$$\varepsilon^2 = -\frac{(E^2 - m_0^2 c^4)a^2}{\hbar^2 c^2} + \frac{l(l+1)C_0}{\alpha^2}, \quad \beta^2 = \frac{2EV_0 a^2}{\hbar^2 c^2} - \frac{l(l+1)C_1}{\alpha^2},$$

$$\gamma^2 = -\frac{V_0^2 a^2}{\hbar^2 c^2} + \frac{l(l+1)C_2}{\alpha^2},$$

with real $\varepsilon \geq 0$ ($E^2 \leq m_0^2 c^4$) for the bound states; β and γ are real and positive.

If to rewrite equation (3.7) by using a new variable of the form

$$z = \left(1 + e^{\frac{r-R_0}{a}} \right)^{-1}, \text{ we obtain}$$

$$u''(z) + \frac{1-2z}{z(1-z)}u'(z) + \frac{-\varepsilon^2 + \beta^2 z - \gamma^2 z^2}{(z(1-z))^2}u(z) = 0, \quad (0 \leq z \leq 1),$$

with $\tilde{\tau}(z) = 1-2z$; $\sigma(z) = z(1-z)$; $\tilde{\sigma}(z) = -\varepsilon^2 + \beta^2 z - \gamma^2 z^2$.

In the NU - method the new function $\pi(z)$ is equal to

$$\pi(z) = \pm \sqrt{\varepsilon^2 + (k - \beta^2)z - (k - \gamma^2)z^2}.$$

The constant parameter k can be found employing the condition that the expression under the square root has a double zero, i.e., its discriminant is equal to zero. So, there are two possible functions for each k

$$\pi(z) = \pm \begin{cases} \left(\varepsilon - \sqrt{\varepsilon^2 - \beta^2 + \gamma^2} \right) z - \varepsilon, & \text{for } k = \beta^2 - 2\varepsilon^2 + 2\varepsilon\sqrt{\varepsilon^2 - \beta^2 + \gamma^2}, \\ \left(\varepsilon + \sqrt{\varepsilon^2 - \beta^2 + \gamma^2} \right) z - \varepsilon, & \text{for } k = \beta^2 - 2\varepsilon^2 - 2\varepsilon\sqrt{\varepsilon^2 - \beta^2 + \gamma^2}. \end{cases}$$

According to the NU - method, from the four possible forms of the polynomial $\pi(z)$ we select on one, for which the function $\tau(z)$ has the negative derivative and root lies in the interval $(0, 1)$. Therefore, the appropriate functions $\pi(z)$ and $\tau(z)$ have the following forms

$$\pi(z) = \varepsilon - \left(\varepsilon + \sqrt{\varepsilon^2 - \beta^2 + \gamma^2} \right) z,$$

$$\tau(z) = 1 + 2\varepsilon - 2 \left(1 + \varepsilon + \sqrt{\varepsilon^2 - \beta^2 + \gamma^2} \right) z$$

and

$$k = \beta^2 - 2\varepsilon^2 - 2\varepsilon\sqrt{\varepsilon^2 - \beta^2 + \gamma^2}.$$

Then, the constant $\lambda = k + \pi'(z)$ takes the following form:

$$\lambda = \beta^2 - 2\varepsilon^2 - 2\varepsilon\sqrt{\varepsilon^2 - \beta^2 + \gamma^2} - \varepsilon - \sqrt{\varepsilon^2 - \beta^2 + \gamma^2}. \quad (3.8)$$

An alternative definition of λ_{n_r} is

$$\lambda = \lambda_{n_r} = 2\left(\varepsilon + \sqrt{\varepsilon^2 - \beta^2 + \gamma^2}\right)n_r + n_r(n_r + 1). \quad (3.9)$$

Having compared Eqs.(3.8) and (3.9) we obtain

$$\varepsilon + \sqrt{\varepsilon^2 - \beta^2 + \gamma^2} + n_r + \frac{1}{2} - \frac{\sqrt{1+4\gamma^2}}{2} = 0$$

or

$$\varepsilon + \sqrt{\varepsilon^2 - \beta^2 + \gamma^2} - n' = 0. \quad (3.10)$$

Here

$$n' = -n_r + \frac{\sqrt{1+4\gamma^2} - 1}{2} \quad (3.11)$$

n_r being the radial quantum number ($n_r = 0, 1, 2, \dots$). From Eq.(3.10), we find

$$\varepsilon = \frac{1}{2} \left(n' + \frac{\beta^2 - \gamma^2}{n'} \right). \quad (3.12)$$

Because for the bound states $\varepsilon \geq 0$, we get

$$n' > 0,$$

and

$$-n'^2 \leq \beta^2 - \gamma^2 \leq n'^2. \quad (3.13)$$

If $n' > 0$, there exist bound states, otherwise there are not bound states at all. By using Eq.(3.11) this relation can be recast into the form

$$0 \leq n_r < \frac{\sqrt{1+4\gamma^2} - 1}{2}, \quad (3.14)$$

i.e. it gives the finite coupling value. After substituting α, γ, C_2 into Eq.(3.14) we get

$$0 \leq n_r < \frac{\sqrt{1 + \frac{192a^4 l(l+1)}{R_0^4} - \frac{4V_0^2 a^2}{\hbar^2 c^2}} - 1}{2}. \quad (3.15)$$

If $-n'^2 \leq \beta^2 - \gamma^2 \leq n'^2$, there exists bound states, otherwise there are not bound states. After substituting the values of β, γ, n' into (3.13) and by using the expressions of α, C_1, C_2 , we obtain

$$\begin{aligned}
& -\frac{V_0}{2} + \frac{4a\hbar^2 c^2 l(l+1)}{R_0^3 V_0} - \frac{\hbar^2 c^2}{2a^2 V_0} \left[-n_r + \frac{\sqrt{1 + \frac{192a^4 l(l+1)}{R_0^4} - \frac{4V_0^2 a^2}{\hbar^2 c^2}} - 1}{2} \right]^2 \leq E_{n,l} \leq -\frac{V_0}{2} + \\
& \frac{4a\hbar^2 c^2 l(l+1)}{R_0^3 V_0} + \frac{\hbar^2 c^2}{2a^2 V_0} \left[-n_r + \frac{\sqrt{1 + \frac{192a^4 l(l+1)}{R_0^4} - \frac{4V_0^2 a^2}{\hbar^2 c^2}} - 1}{2} \right]^2.
\end{aligned} \tag{3.16}$$

The condition $\gamma > 0$ gives the definite coupling value for the potential depth V_0

$$0 < V_0 < \frac{4\hbar c a \sqrt{3l(l+1)}}{R_0^2}. \tag{3.17}$$

Thus, substituting the values of ε , β , γ , n' into Eq.(3.12) and by using the expressions of α, C_0, C_1, C_2 , one can find the energy eigenvalues $E_{n,l}$

$$\begin{aligned}
E_{n,l} = & -\frac{V_0}{2} \left(1 - \frac{32a^3 l(l+1)}{R_0^3 \left[\left(\sqrt{1 + \frac{192a^4 l(l+1)}{R_0^4} - \frac{4V_0^2 a^2}{\hbar^2 c^2}} - 2n_r - 1 \right)^2 + \frac{4V_0^2 a^2}{\hbar^2 c^2} \right]} \right) \left\{ 1 \mp \frac{2c}{V_0} \times \right. \\
& \left. \left(\sqrt{1 + \frac{192a^4 l(l+1)}{R_0^4} - \frac{4V_0^2 a^2}{\hbar^2 c^2}} - 2n_r - 1 \right) \left[\left(\sqrt{1 + \frac{192a^4 l(l+1)}{R_0^4} - \frac{4V_0^2 a^2}{\hbar^2 c^2}} - 2n_r - 1 \right)^2 + \frac{4V_0^2 a^2}{\hbar^2 c^2} \right] \right\} \times (3.18) \\
& \left(n_0^2 c^2 + \frac{\hbar^2 l(l+1)}{R_0^2} \left(1 - \frac{4a}{R_0} + \frac{12a^2}{R_0^2} \right) \right) \left[\left(\sqrt{1 + \frac{192a^4 l(l+1)}{R_0^4} - \frac{4V_0^2 a^2}{\hbar^2 c^2}} - 2n_r - 1 \right)^2 + \frac{4V_0^2 a^2}{\hbar^2 c^2} - \right. \\
& \left. \frac{32a^3 l(l+1)}{R_0^3} \right]^2 - \frac{\hbar^2}{16a^2} \left. \right\}^{\frac{1}{2}}.
\end{aligned}$$

If all three conditions (3.15), (3.16) and (3.17) are satisfied simultaneously, the bound states exist. From Eq.(3.15) is seen that if $l = 0$, then one gets $n_r < 0$. Hence, the Klein - Gordon equation for the standard Woods - Saxon potential with zero angular momentum does not lead to bound states. For larger values of V_0 $\left(V_0 > \frac{4\hbar c a \sqrt{3l(l+1)}}{R_0^2} \right)$ the condition (3.15) is not satisfied. Therefore, no bound states exist for these values of V_0 .

According to Eq. (3.18) the energy eigenvalues depend on the depth of the potential V_0 , the width of the potential R_0 , and the surface thickness a . Any energy eigenvalue must be less than V_0 . If constraints imposed on n_r , V_0 and $E_{n_r,l}$ are satisfied, the bound states appear. From Eq. (3.17) is seen that the potential depth increases, when the parameter a increases, but the parameter R_0 is decreasing and vice versa. Therefore, one can say that the bound states exist within this potential. The energy spectrum Eq. (3.18) are limited, i.e. we have only the finite number of energy eigenvalues.

In addition, we have seen that there are some restrictions on the potential parameters in order to obtain bound state solutions. We also point out that the exact results obtained for the standard Woods - Saxon potential may have some interesting applications for studying different quantum mechanical and nuclear scattering problems.

Now, we are going to determine the radial eigenfunctions of this potential. Having substituted $\pi(z)$ and $\sigma(z)$ into Eq.(2.4) and then solving first order differential equation, one can find the finite function $\Phi(z)$ in the interval $[0, 1]$

$$\Phi(z) = z^\varepsilon (1-z)^{\sqrt{\varepsilon^2 - \beta^2 + \gamma^2}}.$$

It is easy to find the second part of the wave function from the definition of the weight function

$$\rho(z) = z^{2\varepsilon} (1-z)^{2\sqrt{\varepsilon^2 - \beta^2 + \gamma^2}}$$

and substituting it into Rodrigues relation (2.4), we get

$$y_{n_r}(z) = B_{n_r} z^{-2\varepsilon} (1-z)^{-2\sqrt{\varepsilon^2 - \beta^2 + \gamma^2}} \frac{d^{n_r}}{dz^{n_r}} \left[z^{n_r + 2\varepsilon} (1-z)^{n_r + 2\sqrt{\varepsilon^2 - \beta^2 + \gamma^2}} \right],$$

where $B_{n_r} = \frac{1}{n_r!}$ is the normalization constant [12]. Then, y_{n_r} is given by the Jacobi polynomials

$$y_{n_r}(z) = P_{n_r}^{(2\varepsilon, 2\sqrt{\varepsilon^2 - \beta^2 + \gamma^2})}(1-2z),$$

where

$$P_n^{(\alpha, \beta)}(1-2z) = \frac{1}{n!} z^{-\alpha} (1-z)^{-\beta} \frac{d^n}{dz^n} \left[z^{n+\alpha} (1-z)^{n+\beta} \right].$$

The corresponding $u_{n_r,l}(z)$ radial wave functions are found to be

$$u_{n_r,l}(z) = C_{n_r,l} z^\varepsilon (1-z)^{\sqrt{\varepsilon^2 - \beta^2 + \gamma^2}} P_{n_r}^{(2\varepsilon, 2\sqrt{\varepsilon^2 - \beta^2 + \gamma^2})}(1-2z),$$

where $C_{n_r,l}$ are the normalization constants determined using $\int_0^1 [u_{n_r,l}(r)]^2 dr = 1$ constraint.

CONCLUSION

In this paper, we have analytically calculated energy eigenvalues of the bound states and corresponding eigenfunctions in the Woods - Saxon potential. The energy eigenvalue expression for Woods - Saxon potential is given by Eq.(3.18). As it should be expected, for any given set of parameters V_0 , R_0 and a , the energy levels of the standard Woods-Saxon potential are positive. The obtained results are interesting for both theoretical and experimental physicists, because they provide exact expression for energy eigenvalues and corresponding eigenfunctions.

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**İXTİYARİ l - HALINDA VUD - SAKSON POTENSİALLI SAHƏDƏ KLEYN
- GORDON TƏNLIYİNİN ANALİTİK HƏLLİ**

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XÜLASƏ

Məqalədə Vud–Sakson potensiallı sahədə radial Kleyn-Gordon tənliyinin analitik həlli verilmişdir. Hesablamalar ixtiyari l halında mərkəzəqaçma potensialına Pекерис yaxınlaşmasında Nikiforov–Uvarov metodunu tətbiq etməklə aparılmışdır. n_r və l kvant ədədlərinin müxtəlif qiymətlərində əlaqəli halların məxsusi enerjiləri və ona uyğun məxsusi funksiyaları tapılmışdır.

Açar sözlər: Kleyn-Gordon tənliyi, Nikiforov-Uvarov metodu, Pекерис yaxınlaşması, dəqiq həllər

**АНАЛИТИЧЕСКИЕ РЕШЕНИЯ УРАВНЕНИЯ
КЛЕЙНА - ГОРДОНА В ПОТЕНЦИАЛЕ ВУДА-САКСОНА
ДЛЯ ПРОИЗВОЛЬНОГО l - СОСТОЯНИЯ**

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РЕЗЮМЕ

В работе дано аналитическое решение уравнения Клейна - Гордона для потенциала Вуда - Саксона. В вычислениях мы применили метод Никифорова - Уварова, используя приближение Пекериса к центробежному потенциалу для произвольного l - состояния. Для частицы получены точные собственные значения энергии и собственные функции связанных состояний при различных значениях квантовых чисел n_r и l .

Ключевые слова: уравнение Клейна - Гордона, метод Никифорова - Уварова, приближение Пекериса, точные решения